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On radicals in lattices

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To the memory of my teacher Professor Andor Kertész

§ 1. Introduction

In [2] B. STENSTRÖM has defined the radical of a complete lattice L as the meet of all dual atoms of L. Furthermore he has studied properties of the radical in several classes of complete modular lattices. Our aim in this note is to generalize some of the theorems of [2] to certain classes of lattices which are not modular in general but preserve some properties of modularity such as M-symmetry, cross-symmetry and the covering property. Applications are given to some classes of AC-lattices. Our main results are Theorems 3.8 and 3.14.

§ 2. Basic notions

The least element and the greatest element of a lattice, if they exist, are denoted by 0 and 1, respectively. In a lattice L, we say that the element $a \in L$ covers the element $b \in L$ and write $b \prec a$ in case $b < a$ and $b \leqq x \leqq a$ implies $x = b$ or $x = a$. In a lattice L with 0 an element $p \in L$ is called an atom, if $0 \prec p$. In a lattice L with 1 an element $m \in L$ is called a dual atom if $m \prec 1$. $a \| b$ means that $a \in L$ and $b \in L$ are incomparable elements, that is neither $a \leqq b$ nor $b \leqq a$; $c = a + b$ stands for $c = a \cup b$ and $a \cap b = 0$; if $a \leqq b$, then $[a, b] = \{x \in L \colon a \leqq x \leqq b\}$.

Definition 2.1. Let L be a lattice and $a, b \in L$. We say that a, b is a *modular pair* and write $(a, b)M$ when $c \leqq b$ implies $(c \cup a) \cap b = c \cup (a \cap b)$ in L. We say that a, b is a *dual modular pair* and we write $(a, b)M^*$ when $c \geqq b$ implies $(c \cap a) \cup b = c \cap (a \cup b)$ in L.

Lemma 2.2 ([1, Lemma 1.3, p. 2]). *Let L be a lattice and $a, b \in L$. If both $(a, b)M$ and $(a, b)M^*$ then the sublattices $[a, a \cup b]$ and $[a \cap b, b]$ are isomorphic and*

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we write $[a, a \cup b] \cong [a \cap b, b]$. *An isomorphism is effected by the following mutual inverse mappings:* $\varphi(x) = x \cup a$ *and* $\psi(y) = y \cap b$.

From the isomorphic mappings of the preceding lemma one gets:

Corollary 2.3. (i) *If* m *is a dual atom of* $[a \cap b, b]$ *then* $\varphi(m) = m \cup a$ *is a dual atom of* $[a, a \cup b]$;

(ii) $\varphi(\bigcap m_\nu) = \bigcap \varphi(m_\nu)$ $(m_\nu \in [a \cap b, b])$ *if the meets exist,*

Proof. (i) is obvious. For (ii) we have

$$\psi[\bigcap \varphi(m_\nu)] = b \cap [\bigcap \varphi(m_\nu)] = \bigcap [b \cap \varphi(m_\nu)] = \bigcap \psi\varphi(m_\nu) = \bigcap m_\nu.$$

Hence

$$\bigcap \varphi(m_\nu) = \varphi\psi[\bigcap \varphi(m_\nu)] = \varphi(\bigcap m_\nu).$$

Lemma 2.4 ([1, Lemma 1.5, p. 2]) *Let* L *be a lattice and* $a, b \in L$. *If* $(a, b)M$ *then* $(a_1, b_1)M$ *for any* $a_1 \in [a \cap b, a]$ *and* $b_1 \in [a \cap b, b]$.

Definition 2.5 A lattice L is called *modular* when $(a, b)M$ holds for all $a, b \in L$. A lattice L is called *M-symmetric* when $(a, b)M$ implies $(b, a)M$ in L. A lattice L is called *cross-symmetric* if $(a, b)M$ implies $(b, a)M^*$ in L.

For a detailed treatment of symmetric lattices we refer to [1].

Theorem 2.6 ([1, Theorem 1.9, p. 3]). *A cross-symmetric lattice is M-symmetric.*

Corollary 2.7. *Let* L *be a cross-symmetric lattice and* $a, b \in L$. *If* $(a, b)M$ *then* $(b, a)M^*$, $(b, a)M$ *and* $(a, b)M^*$.

Proof. The assertion follows immediately from Definition 2.5 and Theorem 2.6.

The implication

$$N^*: a \prec a \cup b \quad \text{implies} \quad a \cap b \prec b$$

plays an important role in this paper. It is a sort of dual covering property and is satisfied in every modular lattice.

A lattice L with 0 is called atomistic if every element of L is the join of a family of atoms. An element of a lattice L with 0 is called a finite element when it is either 0 or the join of a finite number of atoms. The set of all finite elements of L is denoted by $F(L)$. The covering property is introduced as follows:

if p is an atom and $a \cap p = 0$, then $a \prec a \cup p$. We call L an AC-lattice if it is an atomistic lattice with covering property.

For the theory of AC-lattices we refer to [1].

Definition 2.8 In a lattice L, an element $a \in L$ is called a *modular element* when $(x, a)M$ for every $x \in L$. A lattice L with 0 is called *finite-modular*, when every finite element of L is a modular element.

Theorem 2.9 ([1, Theorem 9.5, p. 42]). *Let L be an AC-lattice. The following two statements are equivalent:*

(i) *L is finite-modular;*

(ii) *in L the implication N^* holds true.*

A lattice L with 0 and 1 is called a *DAC-lattice* when both L and its dual are AC-lattices (cf. [1, p. 123]).

Theorem 2.10 ([1, Theorem 27.6, p. 123]). *Every DAC-lattice is finite-modular and M-symmetric.*

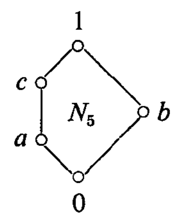
A matroid lattice may be defined as an upper continuous AC-lattice (cf. [1, p. 56]). Now we are ready to define the radical and to study its properties.

§ 3. The radical and its properties

In this paragraph L denotes always a complete lattice.

Definition 3.1. Let $[a, b]$ be an interval of a lattice L. The *radical* of $[a, b]$ is the meet of all dual atoms of $[a, b]$, and is denoted by $R[a, b]$. If $[a, b]$ has no dual atom, then $R[a, b]=b$. Instead of $R[0, 1]$ we shall write $R(L)$. A lattice L is called *radical free* if $R(L)=0$. A lattice L is called *strongly radical free* if $R[a, 1]=a$ for every $a \in L$.

By definition, a strongly radical free lattice is radical free. The converse is not true; consider the following lattice



We have $R(N_5)=b \cap c=0$ but $R[a, 1]=c>a$.

The following lemma gives an equivalent condition for an AC-lattice to be strongly radical free.

Lemma 3.2 *An AC-lattice L is strongly radical free if and only if L is relatively dually atomic (that is, for every $a>b$ there exists a dual atom m of L such that $a>a \cap m \geqq b$).*

Proof. By definition, L is strongly radical free if and only if every $a \in L$ is the meet of those dual atoms m_ν of L, for which $m_\nu \geqq a$ (in the terminology of [1]:

if and only if L is dually atomistic). By the dual of [1, Lemma 7.2, p. 30] this is the case if and only if L is relatively dually atomic.

Corollary 3.3. *Any DAC-lattice is strongly radical free.*

Proof. A *DAC*-lattice is by definition relatively dually atomic, Hence the assertion follows from the preceding lemma.

Corollary 3.4 *Any matroid lattice is strongly radical free.*

Proof. A matroid lattice is relatively dually atomic (cf. [1, Remark 13. 1, p. 56]). Hence the assertion follows from Lemma 3.2.

Corollary 3.5 ([2, Proposition 12]). *If L is a modular matroid lattice then* $R(L)=0$.

Now we are going on to study properties of the radical in lattices which need not be modular but satisfy certain conditions that are fulfilled in the presence of modularity.

Lemma 3.6. *Let L be a lattice in which N^* is satisfied. If $a \leqq b_1 \leqq b_2$ in L then*

$$R[a, b_1] \leqq R[a, b_2].$$

Proof. Let first $m \in [a, b_2]$ and $m \prec b_2$. If $b_1 \leqq m$, then $R[a, b_1] \leqq b_1 \leqq m$. Let now $b_1 \| m$. Then $b_1 \cup m = b_2 \succ m$ and hence by N^* one has $b_1 \cap m \prec b_1$. From this it follows that $R[a, b_1] \leqq b_1 \cap m \leqq m$. Therefore in any case $R[a, b_1] \leqq R[a, b_2]$. If $[a, b_1]$ or $[a, b_2]$ has no dual atoms then the assertion is obvious.

Corollary 3.7 ([2, Proposition 10]). *If L is a modular lattice and $a \in L$, then* $R[0, a] \leqq R(L)$.

Proof. N^* holds in every modular lattice. Applying the preceding lemma, we get the assertion.

It has been proved in [2, Proposition 11] that if in a modular lattice L, 1 is the direct sum of two elements, then $R(L)$ is the direct sum of the radicals of the two direct summands. This is generalized in the following

Theorem 3.8 *Let L be a lattice and let $a, b \in L$. Assume that the following three conditions are satisfied:*

(i) *N^* holds in L;*

(ii) *$(a, b)M$, $(b, a)M^*$ and $(b, a)M$, $(a, b)M^*$ hold in L;*

(iii) *$(b, R[a \cap b, a])M^*$ or $(a, R[a \cap b, b])M^*$ holds in L.*

Then

$$R[a \cap b, a] \cup R[a \cap b, b] = R[a \cap b, a \cup b].$$

Proof. Let L be a lattice and $a, b \in L$. Let (i), (ii) and

$$(b, R[a \cap b, a])M^* \tag{1}$$

be satisfied in L. If instead of (1) the relation $(a, R[a \cap b, b])M^*$ is satisfied then the proof is similar to the now given one. By Lemma 3.6 we get

$$R[a \cap b, a] \cup R[a \cap b, b] \leqq R[a \cap b, a \cup b]. \tag{2}$$

Now we prove that the converse inequality holds true in L. By condition (ii) we get from lemma 2.2 the following isomorphisms:

$$[a, a \cup b] \cong [a \cap b, b] \quad \text{and} \quad [b, a \cup b] \cong [a \cap b, a].$$

Let $\varphi(x) = x \cup a$ and $\bar{\varphi}(x) = x \cup b$. By $\{m_\alpha : \alpha \in A\}$ we denote the set of the dual atoms of $[a \cap b, b]$ and by $\{n_\beta : \beta \in B\}$ we denote the set of the dual atoms of $[a \cap b, a]$. Then we have by Corollary 2.3

$$\varphi(R[a \cap b, b]) = \varphi(\cap m_\alpha) = \cap \varphi(m_\alpha) = \cap (m_\alpha \cup a) \geqq R[a \cap b, a \cup b] \tag{3}$$

and in a similar manner

$$\bar{\varphi}(R[a \cap b, a]) = \bar{\varphi}(\cap n_\beta) = \cap \bar{\varphi}(n_\beta) = \cap (n_\beta \cup b) \geqq R[a \cap b, a \cup b]. \tag{4}$$

By (3) and (4) it follows that

$$R[a \cap b, b] \cup a, \quad R[a \cap b, a] \cup b \geqq R[a \cap b, a \cup b]$$

and hence

$$R[a \cap b, a \cup b] \leqq (R[a \cap b, b] \cup a) \cap (R[a \cap b, a] \cup b). \tag{5}$$

From (1) and from $R[a \cap b, b] \cup a \geqq R[a \cap b, a]$ we get (cf. Definition 2.1)

$$(R[a \cap b, b] \cup a) \cap (b \cup R[a \cap b, a]) = \{(R[a \cap b, b] \cup a) \cap b\} \cup R[a \cap b, a]. \tag{6}$$

Since $R[a \cap b, b] \leqq b$ and $(a, b)M$ we get further (cf. Definition 2.1)

$$(R[a \cap b, b] \cup a) \cap b = R[a \cap b, b] \cup (a \cap b). \tag{7}$$

Now by (5), (6) and (7) it follows that

$$R[a \cap b, a \cup b] \leqq R[a \cap b, a] \cup R[a \cap b, b] \cup (a \cap b). \tag{8}$$

(2) and (8) together prove the theorem.

Corollary 3.9. *Let L be a cross-symmetric lattice in which N^* is satisfied. If $(a, b)M$ then*

$$R[a \cap b, a] \cup R[a \cap b, b] = R[a \cap b, a \cup b].$$

Proof. We show that conditions (i)—(iii) of Theorem 3.8 are satisfied. Condition (i) is satisfied by assumption. Condition (ii) holds by Corollary 2.7. Since $(a, b)M$

and $a \cap b \leqq R[a \cap b, a] \leqq a$ one gets $(R[a \cap b, a], b)M$ by Lemma 2.4. From this it follows that (b, $R[a \cap b, a])M^*$ holds since L is cross-symmetric. This means that condition (iii) is likewise satisfied. Hence the assertion follows from the preceding theorem.

Corollary 3.10. *Let L be a modular lattice and $a, b \in L$. Then*

$$R[a \cap b, a] \cup R[a \cap b, b] = R[a \cap b, a \cup b].$$

Proof. A modular lattice is cross-symmetric and satisfies N^*. Furthermore $(a, b)M$ for all $a, b \in L$. Applying Corollary 3.9, we get the assertion.

Remark. By similar arguments as in Theorem 3.8 we are able to prove the following

Theorem. *Let L be a modular lattice and $a, b \in L$. Then*

$$R[a, a \cup b] \cap R([b, a \cup b] = R[a \cap b, a \cup b].$$

Specializing Corollary 3.10 we get

Corollary 3.11 ([2, Proposition 11]). *If L is a modular lattice and $1 = a + b$ then $R(L) = R[0, a] + R[0, b]$.*

Proof. From $1 = a + b$ we get $R[a \cap b, a] = R[0, a]$, $R[a \cap b, b] = R[0, b]$ and $R[a \cap b, a \cup b] = R[0, 1] = R(L)$. Since $0 \leqq R[0, a] \leqq a$ and $0 \leqq R[0, b] \leqq b$, we have $R[0, a] \cap R[0, b] = 0$. Now the assertion follows from Corollary 3.10.

In the following two corollaries we apply Theorem 3.8 to finite-modular AC-lattices.

Corollary 3.12. *Let L be a finite-modular AC-lattice and let $a, b \in L$. If $a \in F(L)$ then $R[a \cap b, b] = R[a \cap b, a \cup b]$. Similarly, if $b \in F(L)$ then $R[a \cap b, a] = R[a \cap b, a \cup b]$.*

Proof. We show that conditions (i)—(iii) of Theorem 3.8 hold in L. Condition (i) is satisfied by Theorem 2.9. Condition (ii) and condition (iii) hold by [1, Corollary 9.4, p. 42]. If now $a \in F(L)$, then $[0, a]$ is a matroid lattice by [1, Lemma 8.10, p. 37.] Hence $R[a \cap b, a] = a \cap b$ by Corollary 3.4. Applying Theorem 3.8 we get $R[a \cap b, b] = R[a \cap b, a \cup b]$. Similarly $R[a \cap b, a] = R[a \cap b, a \cup b]$ if $b \in F(L)$.

Corollary 3.13. *Let L be a finite-modular AC-lattice and let $a \in F(L)$. If $a \in F(L)$ has a complement in L then $a \cap R(L) = 0$.*

Proof. Let b be a complement of $a \in F(L)$. Since a$\in F(L)$, $a \cap b = 0$ and $a \cup b = 1$

we get by Corollary 3.12 that $R(L)=R[0, b]$. From this it follows that

$$a\cap R(L) = a\cap R[0, b] = a\cap b = 0,$$

which proves the corollary.

Now we put the question: under which conditions can we prove in the preceding corollary the converse implication? An answer to this question is given in

Theorem 3.14. *Let L be an M-symmetric finite-modular AC-lattice and let $a\in F(L)$. If $a\cap R(L)=0$, then $a\in F(L)$ has a complement in L.*

Proof. Let

$$a\cap R(L) = 0. \tag{9}$$

Assume that $a\in F(L)$ has no complement in L. From this assumption we shall derive a contradiction. Let $a_m(\leqq a)$ be a minimal element without complement in L. Such an a_m exists since $a\in F(L)$. Furthermore $a_m\neq 0$, since 0 has the complement 1 in L. From (9) it follows that

$$a_m\cap R(L) = 0.$$

Hence there exists a dual atom $n\in L$ such that $a_m\| n$. Then by N^* (cf. Theorem 2.9)

$$a_m\cap n \prec a_m. \tag{10}$$

By the minimality of a_m, it follows from (10) that $a_m\cap n$ has a complement $b\in L$. Let

$$d \overset{\text{def}}{=} b\cap n.$$

We show that d is a complement of a_m which is a contradiction to our assumption. Evidently

$$n\cap b < b \tag{11}$$

since from $n\cap b=b$ it follows that $b\leqq n$ and $1=(a_m\cap n)\cup b\leqq n$, a contradiction.

From (11) we get by N^* that $n\cap b\prec b$. By [1, Lemma 7.5, p. 31] it follows that $(n, b)M$. Since L is M-symmetric, we get $(b, n)M$. This means that

(12) $x\leqq n$ implies $(x\cup b)\cap n=x\cup(b\cap n)$ in L (cf. Definition 2.1).

Since $a_m\cap n\leqq n$, it follows from (12) that

$$\{(a_m\cap n)\cup b\}\cap n = (a_m\cap n)\cup(b\cap n). \tag{13}$$

Then by the definition of d and by (13)

$$a_m\cup d = a_m\cup (a_m\cap n)\cup d = a_m\cup [(a_m\cap n)\cup(b\cap n)] = a_m\cup[n\cap\{(a_m\cap n)\cup b\}] =$$
$$= a_m\cup(n\cap 1) = a_m\cup n = 1.$$

Furthermore

$$a_m \cap d = a_m \cap n \cap b = 0.$$

Hence d is a complement of a_m. This contradiction proves the theorem.

Corollary 3.15. *Let L be a DAC-lattice. If $a \in F(L)$ then it has a complement in L.*

Proof. A DAC-lattice is finite-modular and M-symmetric by Theorem 2.10. For a DAC-lattice L it follows by Corollary 3.3 that $R(L)=0$. Applying Theorem 3.14 we get the assertion.

We remark that Corollary 3.15 forms a part of [1, Theorem 27.10, p. 124]. Summarizing Corollary 3.13 and Theorem 3.14 we have

Corollary 3.16. *Let L be an M-symmetric finite-modular AC-lattice and let $a \in F(L)$. Then $a \cap R(L)=0$ if and only if $a \in F(L)$ has a complement in L.*

We conclude this paragraph by remarking that the preceding corollary is an extension of a part of [2, Theorem 14].

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